Unit - 2

Application of partial Differential Equation

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Some important second-order PDE

(a).

One-dimensional wave Equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

One dimensional heat Equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Two dimensional laplace Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Two dimensional wave Equation: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

Topic:- classification of LPDE of second order.

Consider $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$ —(1)

(1) elliptic if $B^2 - 4AC < 0$

(2) hyperbolic if $B^2 - 4AC > 0$

(3) parabolic if $B^2 - 4AC = 0$

Note:- (1) If A, B, & C in (1) are constant, then nature of (1) will be the same in whole region i.e for all values of x & y.

(2) If A, B & C are funtion of x & y, then Nature of equation (1) will not the same in whole region i.e for all values of x & y.

Q.1. (a). $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t}$

Here $A = 1$, $B = 1$, $C = 1$

$B^2 - 4AC = 1 - 4 = -3 < 0$, elliptic

For students:- (b). $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

(c) $5\frac{\partial^2 u}{\partial x^2} + 9\frac{\partial^2 u}{\partial x \partial t} + 5\frac{\partial^2 u}{\partial y^2}$.

Q. Find whether the following operators are hyperbolic parabolic and elliptic.

(i). $t\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$.

Here $A = t$, $B = 2$, $C = x$

for hyperbolic if $4 - 4tx > 0$ i.e $tx < 1$

for Parabolic if $4 - 4tx = 0$, i.e $tx = 1$

for elliptic if $4 - 4tx < 0$. i.e; $tx > 1$

Q. 3. Classify the equation

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + (1-y^2)\frac{\partial^2 u}{\partial y^2} - 2y = 0$$

Here, $A = (1-x^2)$, $B = -2xy$, $C = (1-y^2)$

Now $B^2 - 4AC = 4x^2y^2 - 4(1-x^2)(1-y^2)$

$$= 4x^2y^2 - 4(1 - y^2 - x^2 + x^2y^2)$$

$$= \cancel{4x^2y^2} - 4 + 4(x^2+y^2) - \cancel{4x^2y^2}$$

$$= -4 + 4(x^2+y^2)$$

for elliptic if $-4 + 4(x^2+y^2) < 0$ i.e; $x^2+y^2 < 1$

for hyperbolic if $-4 + 4(x^2+y^2) > 0$. i.e $x^2+y^2 > 1$

for ~~elliptic~~ Parabolic if $-4 + 4(x^2+y^2) = 0 \Rightarrow x^2+y^2 = 1$.

Method of separation of variables

~~Consider~~ Let $u(x_1, x_2, x_3 \cdots x_n) = X_1 \cdot X_2 \cdot X_3 \cdots X_n$.

~~dependent~~ dependent variable — independent variable

Here X_1 in terms of x_1
X_2 " " " x_2
⋮
X_n " " " x_n

Note:- PDE $\longrightarrow$ ODE

Q.1. Use method of separation of variables to solve the equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{given that} \quad u(x,0) = 6e^{-3x}$$

Sol^n:- The given equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{——} (1)$$

Let $u = XT$ —— (2) where X in terms of x, T " " t

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx}, \quad \frac{\partial u}{\partial t} = X\frac{du}{dt}$$

Equation (1).

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

$$TX' = 2XT' + XT$$

$$TX' = X(2T' + T)$$

$$\frac{X'}{X} = \frac{2T'}{T} + 1 = -p^2 \quad (\text{say})$$

Case I. $\frac{dX}{dx} + p^2 X = 0$

$$\int \frac{dX}{X} = -p^2 \int dx \Rightarrow \quad X = c_1 e^{-p^2 x}$$

Case II $\frac{2T'}{T} = -(p^2+1) \Rightarrow \log T = -\frac{(p^2+1)}{2}t + \log c_2$

$$T = c_2 e^{-\left(\frac{p^2+1}{2}\right)t}.$$

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from equation (2).

$$u = XT$$

$$u(x,t) = c_1 c_2 e^{-p^2 x - \left(\frac{p^2+1}{2}\right) t} \quad \text{—— (3)}$$

given condition $u(x,0) = 6e^{-3x}$

$$u(x,0) = c_1 c_2 e^{-p^2 x} = 6e^{-3x}$$

$$\Rightarrow p^2 = -3 \quad \& \quad c_1 c_2 = 6$$

Hence required solution is

$$\boxed{u(x,t) = 6e^{-3x-2t}}$$

Example:- Use the method of separation of variables to solve the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{—(1)}$$

Sol^{n} The given equation q

$$u = X(x)\, Y(y)$$

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y} = -p^2 \text{ (say)}$$

(I). $\frac{X''}{X} = -p^2$

$$\frac{d^2X}{dx^2} + p^2 X = 0$$

A.E $m^2 + p^2 = 0$

$$m = \pm ip$$

$X = C.F = c_1 \cos px + c_2 \sin px$

(II) $\frac{Y'}{Y} + 2 = -p^2$

$$\frac{Y'}{Y} = -(p^2+2)$$

$$\log Y = -(p^2+2)\, y + \log c_3$$

$$Y = c_3 \cdot e^{-(p^2+2)y}$$

$$u(x,y) = (A_1 \cdot \cos px + A_2 \sin px)\, e^{-(p^2+2)y}.$$

Q.3. Solve by the method of separation of variables

$$4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \quad u = 3e^{-x} - e^{-5x} \text{ when } t = 0.$$

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Solⁿ The given equation $4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u$ —— (1)

Let $u = X(x) \cdot T(t)$ —— (2)

from (1)

$$4XT' + TX' = 3XT$$

$$\frac{4T'}{T} + \frac{X'}{X} = 3$$

$$\frac{4T'}{T} - 3 = -\frac{X'}{X} = p^2 \quad (\text{say})$$

(i) $\frac{4T'}{T} = p^2 + 3$

$$\frac{dT}{T} = \left(\frac{3+p^2}{4}\right)dt \Rightarrow T = c_1 e^{\left(\frac{3+p^2}{4}\right)t}.$$

(ii). $\frac{-X'}{X} = p^2 \Rightarrow \frac{dX}{X} = -p^2 dx$

$$X = c_2 e^{-p^2 x}$$

$$u(x,t) = c_1c_2\, e^{-p^2x + \left(\frac{3+p^2}{4}\right)t}.$$

$$= b_n e^{-p^2x + \left(\frac{3+p^2}{4}\right)t} \quad \text{where } b_n = c_1c_2$$

most general solution $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-p^2x + \left(\frac{3+p^2}{4}\right)t}$

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{-p^2x} = 3e^{-x} - e^{-5x}$$

$$\Rightarrow b_1 = 3, \quad +p^2 = +1, \quad \& \quad b_2 = -1.$$

$$p^2 = 5$$

$$u(x,t) = 3e^{-x+t} - e^{-5x+2t}.$$

Q. Solve the PDE by separation of variables method

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$$u_{xx} = u_y + 2u; \quad u(0,y) = 0$$

$$\frac{\partial}{\partial x} u(0,y) = 1 + e^{-3y}$$

Solⁿ: The given equation

$$u_{xx} = u_y + 2u \quad \text{—— (1)}$$

Let $u(x,y) = X(x) \cdot Y(y)$ —— (2)

from (1) & (2)

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = K \quad (\text{Say})$$

(1)
$$\frac{X''}{X} = K$$

$$X'' - XK = 0$$

Auxiliary Equation

$$m^2 - K = 0$$

$$m = \pm\sqrt{K}$$

$$X = C.F = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$$

(2)
$$\frac{Y'}{Y} + 2 = K$$

$$\frac{dY}{Y} = (K-2)\,dy$$

$$Y = c_3 e^{(K-2)y}$$

from (2)

$$u(x,t) = \left(c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}\right) c_3 e^{(K-2)y}$$

Applying condition $u(0,y) = 0$

$$c_1 + c_2 = 0, \quad \Rightarrow \quad c_2 = -c_1$$

Thus, $u(x,y) = c_1 c_3 \left(e^{\sqrt{K}x} - e^{-\sqrt{K}x}\right) e^{(K-2)y}$ —— (3)

Most general solution

$$u(x,y) = \sum b_n \left(e^{\sqrt{K}x} - e^{-\sqrt{K}x}\right) e^{(K-2)y}$$

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$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = \sum b_n \sqrt{k} \left(e^{\sqrt{k}x} + e^{-\sqrt{k}x}\right) e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = \sum \underline{b_n \sqrt{k}(2)}\, e^{(k-2)y} = 1 + e^{-3y}.$$

Case (1) $\cancel{b_1 = 1}$ $k-2 = 0 \Rightarrow k = 2,$

$b_1 \sqrt{k}(2) = 1. \quad b_1 = 1/2\sqrt{2}$

Case II $\cancel{b_1 = 1}$, $k-2 = -3$

$k = -1$

$b_2 \sqrt{-1}\,(2) = 1 \Rightarrow b_2 = \frac{1}{2i}$

from (B)

$$u(x,y) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)e^{-3y}$$

$$= \frac{1}{\sqrt{2}} \sinh\sqrt{2}x + e^{-3y} \sin x$$

One Dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ 9

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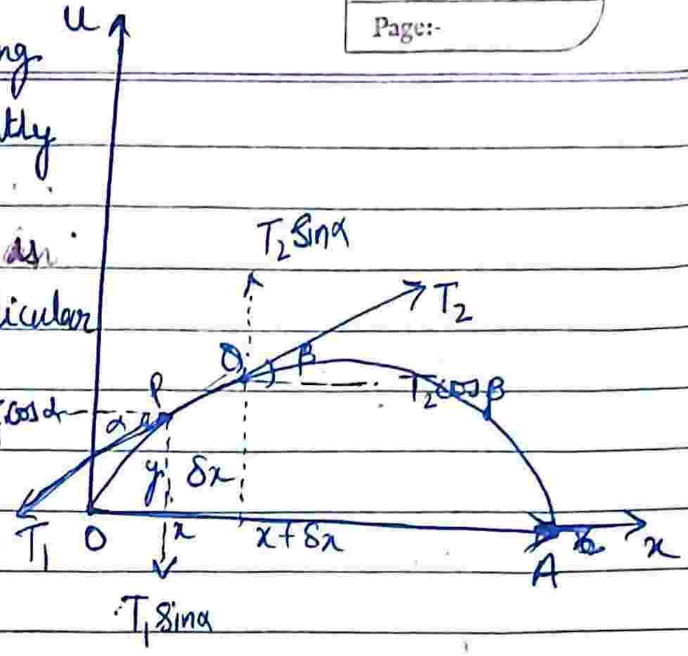
Consider a Uniform elastic string of length l. stretched tightly between O & A.

Taking O as origion & OA as the x-axis, & a perpendicular line through O as the y-axis,

we then distort the string, and at some instant call it $t=0$ we release it and allow it to vibrate



Physical Assumption

① The mass of the string per unit is constant

$\frac{m}{l}$ = constant (where m be the mass per unit)

② Tension $(T) >> W = mg$.

③. Motion is contraint in xy-plane.

~~Consider the motion of an element of length δs~~

The tension T_1 at P & T_2 at Q,

Since there is no motion in the horizontal direction

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant)} \quad — ①$$

By Newtons second low of motion

$$Ma = F$$

$$m\delta x \frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \quad — ②$$

where $M = m\delta x$

①&② $$\frac{m\delta x}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta x}\left(\tan\beta - \tan\alpha\right)$$

$$= \frac{T}{m}\left(\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} + \left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \quad \text{where } c^2 = \frac{T}{m} \qquad (1)$$

This is PDE giving the transverse vibrations of the string.

it is also called one dimensional wave equation.

Boundary Condition

$y(0,t) = 0, \quad y(L,t) = 0 \qquad t \geq 0.$

initial Condition

$y(x,0) = f(x); \quad y_t(x,0) = \cancel{g(x)}\, 0, \quad 0 \leq x \leq L$

Solution of the wave equation

The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ — (1)

Let $y = X(x)\cdot T(t)$

$$XT'' = c^2X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K$$

$$\Rightarrow X'' - KX = 0 \quad \& \quad T'' - c^2K = 0$$

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① When K is positive $\Rightarrow K = p^2$

$$X'' - p^2 X = 0$$

A.E. $m^2 - p^2 = 0$

$m = \pm p.$

$X = C.F = c_1 e^{px} + c_2 e^{-px}$, & similarly $T = c_3 e^{cpt} + c_4 e^{-cpt}$

Thus. $y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$ —— Ⓐ

② when K is ~~positive~~ Negative $\Rightarrow K = -p^2$

$X = c_1 \cos px + c_2 \sin px$ & $T = c_3 \cos cpt + c_4 \sin cpt$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—— Ⓑ}$$

③ When K is zero $\Rightarrow K = 0$

$X = c_1 x + c_2$; $T = c_3 t + c_4.$

$y = (c_1 x + c_2)(c_3 t + c_4)$ —— Ⓒ

Here Ⓐ, Ⓑ & Ⓒ are solutions of ①. ~~Therefore~~ The y must be a periodic function of x and t. therefore the solution must involve trigonometric terms

So,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

is the only suitable solution —— ②

Applying boundary condition, $u(0,t) = 0$, , $u(l,t) = 0$.

$$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow c_1 = 0.$$

$$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt)$$

$\Rightarrow$ Put $c_1 = 0$

$$c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\cancel{c_2} \sin pl = 0 \Rightarrow \sin pl = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

Thus, from (2)

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$$y(x,t) = c_2\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$= \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

where $c_2 c_3 = a_n$ & $c_2 c_4 = b_n$

Adding up the solution for different values of n.

$$y = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}. \quad \text{--- (5)}$$

Now initial condition $y(x,0) = f(x)$, $y_t(x,0) = 0$

$\cancel{\frac{dy(x,0)}{dt} = 0}$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

which is represent Fourier series of $f(x)$

then

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

Now using $y_t(x,0) = 0$

$$0 = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l}\right) b_n \sin\frac{n\pi x}{l}.$$

$$\Rightarrow b_n = 0.$$

from (5) $\quad y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}.$

where $a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$. $\left\{\text{Here } y(x,0) \text{ is known}\right.$

Example based on one-dimensional wave equation :- (4)

Example 1 A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}.$$

Sol^n The equation of the string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— ①

Boundary Condition

$$y(0,t) = 0, \qquad y(l,t) = 0 \quad \text{—— ②}$$

Initial Condition

$$y(x,0) = A \sin \frac{\pi x}{l}, \qquad y_t(x,0) = 0. \quad \text{—— ③}$$

(Given) (Since the string is released from rest hence its initial velocity will be zero)

Let $y = XT$, $\frac{\partial y}{\partial t} = X \frac{dT}{dt}$

$\frac{\partial^2 y}{\partial t^2} = XT''$ Similarly $\frac{\partial^2 y}{\partial x^2} = TX''$

Substituting into ①.

$$XT'' = c^2 T X'' \Rightarrow \frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} = K.$$

Case I when K is positive, $K = p^2$

$$X'' = p^2 X \Rightarrow X'' - p^2 X = 0$$

A.E $m^2 - p^2 = 0$

$m = \pm cp$

$$X = c_1 e^{(px)} + c_2 e^{-(px)}$$

Similarly $\therefore$ $T'' = c_3 e^{cpt} + c_4 e^{-cpt}$

Thus, $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$ — (A) (12)

<u>Case II</u> when k is Negative $\Rightarrow$ $k = -p^2$

$$X'' = -p^2 X \Rightarrow X'' + p^2 X = 0$$

A.E : $m^2 + p^2 = 0$

$m = \pm ip$

$$X = (c_5 \cos px + c_6 \sin px)$$

Similarly $T = c_7 \cos pct + c_8 \cos pct$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct) \quad \text{— (B)}$$

<u>Case III</u> when k is zero, $k = 0$

$$X'' = 0 \Rightarrow X' = c_9$$

$$X = c_9 x + c_{10}$$

Similarly $T = c_{11} t + c_{12}$

$$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad \text{— (C)}$$

Since we are dealing with problem on vibrations the solution must contain periodic funtion. Hence the solution which contains trigonometric terms must be the required solution.

Therefore the general solution of one-dim- wave equation is given by.

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)$$

<u>Applying Boundary condition</u>

$y(0,t) = 0, \quad y(l,t) = 0$

$$c_5(c_7 \cos cpt + c_8 \sin cpt) = 0 \Rightarrow c_5 = 0$$

$$\Rightarrow y(x,t) = \cancel{C_3 \cos px +} (C_4 \sin px)(C_7 \cos pct + C_8 \sin pct)$$

$$y(l,t) = C_6 \sin lx \,(C_7 \cos pct + C_8 \sin pct) \quad (B)$$

$C_6 \neq 0$, $\sin lp = 0 \Rightarrow \sin lp = \sin n\pi$.

$$p = \frac{n\pi}{l} \qquad n \in \mathbb{Z}$$

$$y(x,t) = \left(C_7 \cos \frac{n\pi ct}{l} + C_8 \sin \frac{n\pi ct}{l}\right) C_6 \sin \frac{n\pi x}{l}$$

Applying initial condition

$$\frac{\partial y}{\partial t} = \left(-C_7 \sin \frac{n\pi ct}{l} + C_8 \cos \frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l} C_6 \sin \frac{n\pi x}{l}$$

at $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi c}{l} \cdot C_8 C_6 \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad C_8 = 0.$$

Become.

$$y(x,t) = C_7 C_6 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$y(x,0) = A \sin \frac{\pi x}{l} = C_7 C_6 \sin \frac{n\pi x}{l}$$

$$\Rightarrow C_7 C_6 = A, \; n = 1$$

~~Hence~~ Therefore $\boxed{y(x,t) = A \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l}}$

H. P.

Example 2. Show that the wave equation $c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ can be solved by method of separation of variables. If the initial displacement and velocity of string stretched between $x=0$ and $x=l$ are given $y = f(x)$ and $\frac{\partial y}{\partial t} = g(x)$, determine constants in the series solution. (TU)

Sol^{n}: The given equations $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ——— ①

let $y = X(x) \times T(t)$. ——— ②

Using ② then ①,

$$XT'' = c^2 TX''$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = K \text{ (say)}$$

Case I when K is positive then $K = p^2$

$$X'' - pX = 0 \Rightarrow \text{A.E. } m^2 - p = 0$$

$$m = \pm p$$

$$C.f = c_1 e^{px} + c_2 e^{-px}$$

$P.I = 0$ Thus, $X = c_1 e^{px} + c_2 e^{-px}$

Similarly $T = c_3 e^{pct} + c_4 e^{-pct}$

from ① $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$ ——— (A)

Case II when K is Negative then $K = -p^2$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)$$ ——— (B)

Case III when K is zero then $K = 0$

$$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}).$$ ——— (C)

There ~~are~~ (A), (B) & (C) are three solutions of (1). (15)

But we are dealing with problem of vibration y must be a periodic function of x & t.

therefore the solution must be contain trigonometric terms.

Hence, A is required solution,

i.e $y(x,t) = (C_5 \cos px + C_6 \sin px)(C_7 \cos pct + C_8 \sin pct)$ —— (3)

Boundary Condition

$$y(0,t) = 0 \quad \& \quad y(l,t) = 0$$

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$$C_5 = 0,$$

$$y(x,t) = C_6 \sin px \,(C_7 \cos pct + C_8 \sin pct)$$

Must general solution

$$y(x,t) = \sum_{n=0}^{\infty} (a_n \cos pct + b_n \sin pct) \sin px$$

$$y(l,t) = 0 = \sum_{n=0}^{\infty} (a_n \cos pct + b_n \sin pct) \sin pl$$

$$\sin pl = 0 \Rightarrow p = \frac{n\pi}{l}$$

$$y(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{nct\pi}{l} + b_n \sin \frac{nct\pi}{l}\right) \sin \frac{n\pi x}{l} \quad \text{—— (4)}$$

Initial Condition

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

when $a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$

from (4)

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{nc\pi}{l}\left(-a_n \sin\frac{nc\pi t}{l} + b_n \cos\frac{nc\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

at $t=0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \frac{nc\pi}{l} b_n \sin\frac{n\pi x}{l}.$$

when $\frac{nc\pi b_n}{\cancel{l}} = \frac{2}{\cancel{l}}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$

$$b_n = \frac{2}{nc\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Hence, final solution is

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

when $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$b_n = \frac{2}{nc\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

<u>Example</u> A string is stretched between two fixed points $(0,0)$ and $(l,0)$ and released at rest from the initial deflection given by

$$f(x) = \begin{cases}\left(\frac{2k}{l}\right)x & ; \quad 0<x<\frac{l}{2}\\ \left(\frac{2k}{l}\right)(l-x) & ; \quad \frac{l}{2}<x<l\end{cases}$$

find the deflection of string at any time

The ~~given~~ wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— ① (17)

Boundary Condition

$$y(0,t) = y(l,t) = 0.$$

Initial Condition

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad y(x,0) = f(x) = \begin{cases} \frac{2kx}{l} & 0<x<\frac{l}{2} \\ \frac{2k(l-x)}{l}, & \frac{l}{2}<x<l \end{cases}$$

Thus required solution is $y(x,t) = \sum_{n=0}^{\infty} a_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}$

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} \frac{2kx}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2k(l-x)}{l} \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4k}{l^2}\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x) \frac{\sin n\pi x}{l}\,dx \qquad \text{ILATE}$$

$$= \frac{4k}{l^2}\left[\left(-x\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} + \left(\frac{\sin n\pi x}{\left(\frac{n\pi}{l}\right)^2}\right)_0^{l/2} + \left(-(l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi}{2} - \left(\frac{l}{n\pi}\right)^2\left[\sin n\pi - \sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence $y(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$,

Example! A tightly stretched violin string of length l and fixed at both ends is plucked at $x = \frac{l}{3}$ and assumes initially the shape of a triangle of hight a. find the displacement y at any distance x and any time t after the string is released from rest. (18)

Sol (1) The given one-dim. wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

The Solution (1).

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$$

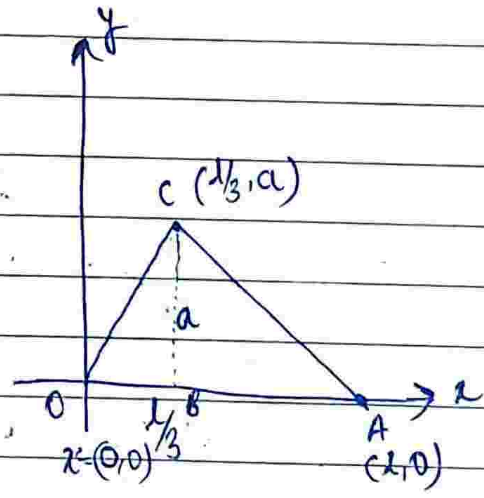
According to Question (ATQ):

Boundary Condition

$$y(0,t) = y(l,t) = 0 \qquad \forall\ t > 0$$

Initial Condition

$$y_1 = 0 + \frac{a-0}{\frac{l}{3}-0}(x-0)$$



$$y_1 = \frac{3a}{l}x$$

& $$y_2 = a + \frac{0-a}{l-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= a + \frac{3a}{2l}\left(x-\frac{l}{3}\right)$$

$$= -\frac{3ax}{2l} + a + \frac{a}{2} = \frac{3a}{2}\left(1-\frac{x}{l}\right)$$

Thus, $$f(x) = y(x,0) = \begin{cases} \frac{3ax}{l} & ; \ 0 < x < \frac{l}{3} \\ \frac{3a}{2}\left(1-\frac{x}{l}\right) & ; \ \frac{l}{3} < x < l \end{cases}$$

Now, Must general Solution

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{l} \frac{3a}{2}\left(1-\frac{x}{l}\right) \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}_0^{l/3}\right.$$

$$\left. + \frac{3a}{l}\left\{\left(1-\frac{x}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \frac{1}{l}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)^2\right\}_{l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l}{n\pi}\cdot\frac{l}{3}\cos\frac{n\pi}{3} + \left(\frac{l}{n\pi}\right)^2\left(\sin\frac{n\pi}{3}\right)\right.$$

$$\left. + \frac{3a}{l}\left(\frac{2}{3}\frac{l}{n\pi}\cos\frac{n\pi}{3} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{3}\right)\right]$$

$$= \cancel{\frac{6a}{n\pi}\left[-\frac{1}{3}\cos\frac{n\pi}{3} + \frac{1}{n\pi}\sin\frac{n\pi}{3}\right] + \frac{6a}{n\pi}\left[\frac{1}{3}\cos\frac{n\pi}{3}\right]}$$

$$+ \frac{3a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$a_n = \frac{9a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

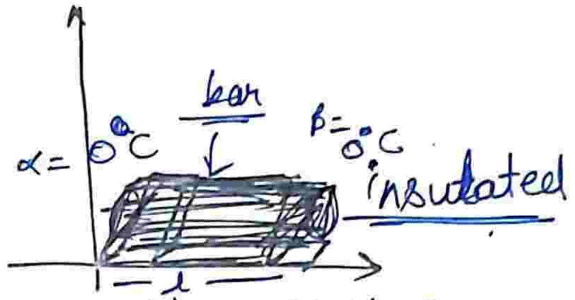
DATE

$$y(x,t) = \frac{9a}{\pi^2} \sum_{n=1}^{\infty} \sin\frac{n\pi}{3} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}.$$

One - Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (1)}$$

where $c^2 = \frac{K}{s\rho}$ is called diffusivity of the material of the bar



$y(0,t) = y(l,t) = 0.$ Some const है

Boundary $y(x,0) = f(x)$.

Note: तीनों change हो सकती है।

Boundary Condition

$$y(0,t) = y(l,t) = 0, \qquad \text{——— (2)}$$

& $y(x,0) = f(x)$ } initial Condition

Solution of Heat equation

The given equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (1)}$$

Let

$$u = X(x) \cdot T(t) \qquad \text{——— (2)}$$

using (2) then,

$$XT' = c^2 T X''$$

$$\frac{X''}{X} = \frac{T'}{c^2 T} = K \quad (\text{say}). \qquad \text{——— (A)}$$

(1) when K is positive then $K = p^2$

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2 p^2 t}, \quad u(x,t) = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t} \qquad \text{——— (3)}$$

(2) when K is Negative then $K = -p^2$

$$X = c_4 \cos px + c_5 \sin px, \quad T = c_6 e^{-c^2 p^2 t}$$

$$u(x,t) = (c_4 \cos px + c_5 \sin px)\, c_6\, e^{-c^2 p^2 t} \qquad \text{——— (4)}$$

(3) when $K = 0$

$$u(x,t) = (c_7 x + c_8)\, c_9 . \qquad \text{——— (5)}$$

These three solutions of (1).

But u decreased at time increase.

$\Rightarrow$ at $t \to \infty$ $\quad u(x,t) = 0$

$$u(x,t) = (c_4 \cos px + c_5 \sin px)\, c_6 e^{-c^2p^2t}$$

is required solution, and other two solutions are rejected.

Boundary condition

$$u(0,t) = u(l,t) = 0.$$

$$u(0,t) = 0 = c_4 c_6 e^{-c^2p^2t} \Rightarrow c_4 = 0.$$

$$u(x,t) = c_5 c_6 \sin px\, e^{-c^2p^2t}$$

$$u(l,t) = c_5 c_6 \sin pl\, e^{-c^2p^2t} = 0$$

$$p = \frac{n\pi}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l}}$$

~~when~~ $\cancel{a_n = \frac{2}{l}\int_0^l f(x)}$

Initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

Hence required solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{c^2n^2\pi^2t}{l}}$$

where $a_n = \frac{2}{l}\int_0^l f(x) \cdot \sin \frac{n\pi x}{l}\, dx$

Example:- A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°C$ and are kept at that temperature. Find the temperature function $u(x,t)$. (23)

Sol (7):- The given Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ①$$

Boundary Conditions

$$u(0,t) = u(l,t) = 0 \quad \forall \ t \geq 0$$

Initial Condition

$$\cancel{u(x,t)} \ u(x,0) = u_0.$$

Solution of ①.

$$u(x,t) = (c_1 \cos px + c_2 \sin px) \, c_3 e^{-c^2p^2t}$$

Boundary Conditions

$$u(0,t) = c_1 = 0$$

$$\Rightarrow u(x,t) = c_2 \sin px \, e^{-c^2p^2t}$$

$$\cancel{u(l,t) =}$$

$$u(l,t) = c_2 c_3 \sin px \, e^{-c^2p^2t}$$

$$\sin px = 0 \Rightarrow \quad p = \frac{n\pi}{l}$$

$$u(x,t) = c_2 c_3 \sin \frac{n\pi}{l} \, e^{-c^2p^2t}$$

Most general Solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} e^{-c^2p^2t}$$

Initial Condition

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

where $a_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left\{ \frac{2u_0}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l \right.$$

$$= \frac{2u_0}{n\pi}\left[-\cos n\pi + 1\right]$$

$$= \begin{cases} 0 & ; \text{ when } n \text{ is even} \\ \frac{4u_0}{n\pi} & ; \text{ when } n \text{ is odd.} \end{cases}$$

Hence

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{c^2 p^2 \pi^2 t}{l^2}}$$

$$\boxed{u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2(2n-1)^2\pi^2 t}{l^2}}}$$

Note:- if $u(0,t) = \alpha$, $u(l,t) = \beta$.

~~Using~~ Convert $u(0,t) = 0$, $u(l,t) = 0$.

(0,α) (l,β)

and $u(x,0) = f(x) = \left[\alpha + \frac{(\beta-\alpha)}{l}x\right]$

Example 2. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C. find the temperature at a distance x from A at time t. Find also the temperature if the change consists of raising the temperature of A to 20°C & reducing that of B to 80°C.

Sol (i) The equation of heat is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— (1) (25)

Here,

Initial temperature distribution is

$$u_1 = 0 + \left(\frac{100-0}{l}\right)x = \frac{100}{l}x$$

and final temperature distribution is

$$u_2 = 0 + \frac{0-0}{l}x = 0$$

Boundary Condition is

$$u(0,t) = u(l,t) = 0 \quad \forall\ t \geq 0 \quad —— (2)$$

and initial Condition is

$$u(x,0) = \cancel{f(x) = u_1 - u_2 =} \frac{100x}{l} \quad —— (3)$$

∴ Solutions of equation (1) is

$$u(x,t) = u_2 + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)e^{-c^2p^2t}$$

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)e^{-c^2p^2t} \quad —— (4)$$

Applying Boundary condition

$$u(0,t) = \sum_{n=1}^{\infty} a_n e^{-c^2p^2t} \Rightarrow a_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin px\, e^{-c^2p^2t}$$

$$u(l,t) = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t} = 0$$

$$\Rightarrow p = \frac{n\pi}{l} \quad \forall n \in I$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

Using initial Condition

$$u(x,0) = \frac{100}{l}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is half-Range sine series. for $\frac{100}{l}x$.

$$b_n = \frac{2}{l}\int_0^l \frac{100}{l} x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[-x \cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right) + \sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2\right]_0^l$$

$$= \frac{200}{l}\left[-l\cos n\pi\cdot\left(\frac{l}{n\pi}\right) + 0\right]$$

$$= \frac{-200}{n\pi}(-1)^n \qquad \left\{ \cos n\pi = (-1)^n \right.$$

Hence the temprature function is

$$\boxed{u(x,t) = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2t}{l^2}}}$$

Second Part II

Initial Condition $\quad u(x,0) = \frac{100x}{l}$

& Boundary Condition are

$$u(0,t) = 20 \quad \& \quad u(l,t) = 80 \quad \forall\, t \geq 0.$$

Final Temperature distribution is

$$u_2 = \alpha + \left(\frac{\beta - \alpha}{l}\right)x$$

$$= 20 + \left(\frac{80-20}{l}\right)x = 20 + \frac{60}{l}x$$

Solution of equation ① is

$$u(x,t) = u_2 + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2p^2t}$$

$$= 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2p^2t}$$

Applying Boundary Condition

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

Using initial condition

$$u(x,0) = \frac{100}{l}x = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{40}{l}x - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l \left(\frac{40}{l}x - 20\right) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{40}{l}x - 20\right)\left(-\cos\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi} + \frac{40}{l}\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi}\right]$$

$$= -\frac{40}{n\pi}(1+\cos n\pi) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

Hence

$$u(x,t) = 20 + \frac{60x}{l} - \frac{80}{\pi}\sum_{n=2,4,6\ldots} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= 20 + \frac{60x}{l} - \frac{40}{\pi}\sum_{m=0}^{\infty} \frac{1}{m}\sin\frac{2m\pi x}{l}\, e^{-\frac{4c^2m^2\pi^2 t}{l^2}}$$

Example:- The ends A and B of a rod of length 20 cm are at temperature 30°c and 80°c until steady state prevails. Then the temperature of the rod ends are changed to 40°c and 60°c respectively. find the temperature distribution function $u(x,t)$.

(20)

Sol^n:- Equation of Heat is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

Initial temperature distribution in rod is

$$u_1 = 30 + \left(\frac{80-30}{20}\right)x = 30 + \frac{5}{2}x$$

& Final Temperature distribution is

$$u_2 = 40 + \left(\frac{60-40}{20}\right)x = 40 + x$$

Boundary Condition (In steady state)

$$u(0,t) = 40 \quad , \quad u(l,t) = 60 \quad \text{—— ②}$$

and Initial Condition is

$$u(x,0) = 30 + \frac{5}{2}x \quad \text{—— ③}$$

Therefore, Solution of ① is

$$u(x,t) = u_2 + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2c^2t}$$

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2c^2t} \quad \text{—— ④}$$

Boundary Condition

{

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{20}\right) e^{-\left(\frac{n\pi}{20}\right)^2 t}$$

Initial condition

$$u(x,0) = 30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

$$\frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{20}\right)$$

$$b_n = \frac{2}{20}\int_0^{20}\left(\frac{3}{2}x - 10\right)\sin\frac{n\pi x}{20}\,dx$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}x - 10\right)\left(-\cos\frac{n\pi x}{20}\right)\cdot\frac{20}{n\pi} + \frac{3}{2}\left(\sin\frac{n\pi x}{20}\right)\left(\frac{20}{n\pi}\right)^2\right]_0^{20}$$

$$= \frac{1}{10}\left[\frac{-20\times 20}{n\pi}\cos n\pi - \frac{200}{n\pi}\right]$$

$$= -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

Therefore

$$u(x,t) = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\left(\frac{2(-1)^n + 1}{n}\right)\sin\frac{n\pi x}{20}\, e^{-\left(\frac{n\pi}{20}\right)^2 t}$$

The given laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{———} \ (1)$$

and Boundary Condition

$u(x,0) = 0, \quad u(x,b) = 0, \quad u(a,y) = 0$

& $u(a,y) = f(x)$ parallel to y-axis

Solution of (1) as:

let $u = X \cdot Y$

$$Y X'' + Y'' X = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = K \quad \text{———} \ (2)$$

(1) when K is positive $k = p^2$

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{———} \ (3)$$

(2) when k is negative $k = -p^2$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{———} \ (4)$$

(3) when k is zero, $k = 0$.

$$u = (c_1 x + c_2)(c_2 y + c_3) \quad \text{———} \ (5)$$

Apply Boundary condition in (3).

$$u(x,0) = (c_1 e^{px} + c_2 e^{-px}) c_3 = 0 \Rightarrow c_3 = 0$$

from (5)

$$u(x,0) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin py \quad \text{———} \ (6)$$

$$u(0,y) = (c_1 + c_2) c_4 \sin py = 0 \Rightarrow c_1 = -c_2$$

from (6)

$$u(x,y) = c_1 (e^{px} - e^{-px}) c_4 \sin py \quad \text{———} \ (7)$$

$$u(x,b) = c_1\left(e^{px} - e^{-px}\right) c_4 \sin pb = 0$$

$$\sin pb = 0 \Rightarrow p = \frac{n\pi}{b}$$

from (7)

$$u(x,y) = c_1\left(e^{n\pi x/b} - e^{-n\pi x/b}\right) c_4 \sin \frac{n\pi y}{b}$$

$$= c_1 \sinh \frac{n\pi x}{b} \, c_4 \sin \frac{n\pi y}{b}$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$

Apply $u(a,y) = f(y)$

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b}$$

$$\Rightarrow a_n = \frac{1}{\sinh \frac{n\pi a}{b}} \cdot \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy$$

$$a_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy$$

$$a_n = \frac{2}{\left(\sinh \frac{n\pi a}{b}\right) b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

Two Dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{where } c^2 = \frac{T}{m}$$

Two Dimensional Heat Flow :

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \frac{k}{s\rho}$ is called diffusivity of the material of the bar

Note I In steady state, $\frac{\partial u}{\partial t} = 0$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

which is called Laplace's equation in two dimensions

Solution of Laplaces equation in two Dimensions.

Laplaces Equation in 2-Dim. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

Let $u = X(x) \cdot Y(y)$ ——— (2)

Using (2). then.

$$YX'' + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = k \quad (\text{Say}) \quad \text{——— (3)}$$

Case I when k is positive, $k = p^2$

$$\frac{X''}{X} = p^2 \Rightarrow X'' - p^2 X = 0$$

A.E. $m^2 - p^2 = 0$

$m = \pm p$

$C.F = (c_1 e^{px} + c_2 e^{-px})$ & $P.I = 0$

$X = (c_1 e^{px} + c_2 e^{-px})$

Similarly $Y = c_3 \cos py + c_4 \sin py$

$u(x,y) = (c_1 e^{px} + c_2 e^{-px}) \times (c_3 \cos py + c_4 \sin py)$ ——— (4)

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Case II When k is Negative ; $k=-p^2$

$$X = (C_5 \cos px + C_6 \sin px) \ \& \ Y = C_7 e^{py} + C_8 e^{-py}$$

$$u(x,y) = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \longrightarrow (B)$$

Case III when k is zero $\Rightarrow k=0$

$$X = C_9 x + C_{10} \ \& \ Y = C_{11} y + C_{12}$$

$$u(x,y) = (C_9 x + C_{10})(C_{11} y + C_{12}) \longrightarrow (C)$$

Here (A), (B) & (C) are three solutions of laplace equation But we have to choose that solution which is consistent with Physical nature of the problem. and given boundary condition.

$\therefore$ Hence (B) is required solution

$$\boxed{u(x,y) = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py})}$$

Example! - Use separation of variables method to solve the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Subject to boundary Condition $u(0,y) = u(l,y) = u(x,0) = 0$

& $u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$

Soln :- The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$$

Let $u = X(x) \cdot T(t) \longrightarrow (2)$

} missing step

$k = p^2$ $u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \longrightarrow (B)$

$k = -p^2$ $u(x,y) = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \longrightarrow (C)$

$k = 0$ $u(x,y) = (C_9 x + C_{10})(C_{11} y + C_{12}) \longrightarrow (A)$

Using condition in (B).

$$u(0,y) = (C_1 + C_2)(C_3 \cos py + C_4 \sin py) = 0$$

$$C_1 = -C_2$$

$\Rightarrow$ ~~u(x,y) = 0~~ ~~imp~~ ~~which is impossible~~

$u(x,y) = C_1($

$u(x,0) = C_1($

$u(x,y) = C_1$

Using Condition in (A) (27)

$$u(0,y) = c_{10}(c_{11}y + c_{12}) = 0 \Rightarrow c_{10} = 0$$

$$u(x,y) = c_9 x(c_{11}y + c_{12})$$

$$u(l,y) = c_9 l(c_{11}y + c_{12}) = 0 \Rightarrow c_9 = 0$$

Thus, $u(x,y) = 0$ which is impossible.

Using Boundary Condition in (C).

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

$$u(0,y) = c_5(c_7 e^{py} + e^{-py}) = 0 \Rightarrow c_5 = 0$$

$$u(x,y) = c_6 \sin px \,(c_7 e^{py} + c_8 e^{-py})$$

$$u(l,y) = c_6 \sin pl \,(c_7 e^{py} + c_8 e^{-py}) = 0$$

$$\sin pl = 0 = \sin n\pi$$

$$p = \frac{n\pi}{l} \quad \forall n \in I$$

$$u(x,y) = \sin\frac{n\pi x}{l}\left(a_n e^{\frac{n\pi y}{l}} + b_n e^{-\frac{n\pi y}{l}}\right)$$

where $a_n = c_6 c_7$

$b_n = c_6 c_8$

$$u(x,0) = \sin\left(\frac{n\pi x}{l}\right)(a_n + b_n) = 0$$

$$a_n = -b_n$$

$$u(x,y) = a_n \sin\left(\frac{n\pi x}{l}\right)\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) \quad \text{——— (3)}$$

$$u(x,a) = \cancel{\sin\frac{n\pi x}{l}} = a_n \cancel{\sin\frac{n\pi x}{l}}\left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right)$$

$$a_n = \frac{1}{e^{n\pi a/l} - e^{-n\pi a/l}} = \frac{1}{2\sinh\left(\frac{n\pi a}{l}\right)}$$

$\ldots^{px} - e_7^{-px})(c_3 \cos py + c_4 \sin py)$	$u(l,y) = c(e^{pl} - e^{-pl})c_4 \sin py = 0$
$\ldots^{px} - e^{-px})\, c_3 = 0 \Rightarrow c_3 = 0$	$\Rightarrow c = 0 \;\&\; c_4 = 0$
$\ldots e^{px} - e^{-px})c_4 \sin py$	$\Rightarrow u(x,y) = 0$ which is impossible.

Hence

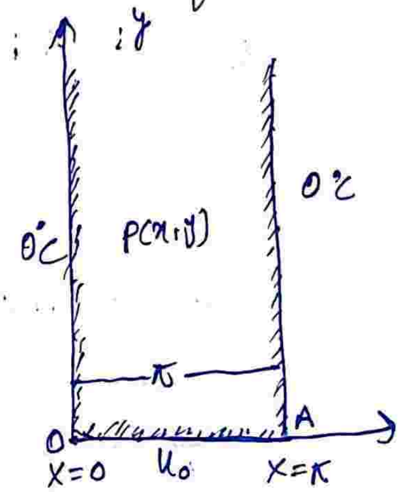
$$u(x,y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{2\sinh\left(\frac{n\pi a}{l}\right)} \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin\frac{n\pi x}{l}.$$

Example: 2. An infinitely long plane uniform plate is bounded by two parallel edge and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

Sol^n: In steady state, two dimensional heat flow equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} ①$$



Boundary condition

$$u(0,y) = 0 = u(\pi, y)$$

$$\lim_{y\to\infty} u(x,y) = 0 \quad (0<x<\pi)$$

$$u(x,0) = u_0 \quad (0<x<\pi)$$

Sol^n of equation ① is

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

$$u(0,y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0 \Rightarrow c_1 = 0$$

$$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py})$$

$$u(\pi,y) = c_2 \sin p\pi (c_3 e^{py} + c_4 e^{-py}) = 0 \Rightarrow p = \frac{n\pi}{\pi}$$

$$p = n$$

$$u(\pi,y) = c_2 \sin n\pi (c_3 e^{ny} + c_4 e^{-ny})$$

$$\lim_{y\to\infty} u(x,y) = 0 = c_2 \sin nx \lim_{y\to\infty} (c_3 e^{ny} + c_4 e^{-ny}) = 0$$

$$\Rightarrow \quad c_3 = 0,$$

$$u(x,y) = c_2 c_4 \, e^{-ny} \sin nx = b_n e^{-ny} \sin nx \qquad \text{where } c_2 c_4 = b_n$$

must general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx$$

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin(nx) \, dx \qquad \underline{\text{By fourier series}}$$

$$= \frac{2}{\pi} u_0 \left[-\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2u_0}{n\pi} \left[-(-1)^n + 1 \right]$$

$$= \begin{cases} \dfrac{4u_0}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$u(x,y) = \frac{4u_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sin nx}{n} e^{-ny}$$

$$\boxed{u(x,y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1)x \; e^{-(2n-1)y}}$$

Example Solve the Laplace eqⁿ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y}=0$ in a rectangle in the xy-plane with

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Page:-

$u(x,0)=0,\quad u(x,b)=0,\quad u(0,y)=0 \;\&\; u(a,y)=f(y)$

Parallel to y-axis

Solⁿ The given equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \longrightarrow (1)$$

let $u=X(x)\cdot Y(y) \longrightarrow (2)$

using (2).

$$YX''+XY''=0$$

$$\frac{X''}{X}=\frac{-Y''}{Y}=k \quad (\text{say}). \longrightarrow (3)$$

Case I. when k is positive; $k=p^2$

$$u(x,y)=(c_3\cos py+c_4\sin py)(c_1e^{px}+c_2e^{-px}) \longrightarrow (A)$$

Case II when k is Negative, $k=-p^2$

$$u(x,y)=(c_5\cos px+c_6\sin px)(c_7e^{py}+c_8e^{-py}) \longrightarrow (B)$$

Case III when k is zero; $k=0$

$$u(x,y)=(c_9x+c_{10})(c_{11}y+c_{12}) \longrightarrow (C)$$

Applying Boundary Condition in (A).

$u(x,0)=0 \Rightarrow c_3=0.$

$$u(x,y)=c_4\sin py\,(c_1e^{px}+c_2e^{-px})$$

$u(0,y)=0,$

$$u(0,y)=c_4\sin py\,(c_1+c_2)=0 \Rightarrow c_1=-c_2$$

$$u(x,y)=2c_1c_4\sin py\,\sinh px$$

$u(x,b)=0,\quad p=\frac{n\pi}{b}$

$$u(x,y)=2c_1c_2\sin\frac{n\pi y}{b}\sinh\left(\frac{n\pi x}{b}\right)$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}$$

Apply $u(a,y) = f(y)$

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi a}{b}$$

$$\Rightarrow a_n = \frac{1}{\sinh\frac{n\pi a}{b}} \cdot \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\, dy$$

$$a_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\, dy$$

$$a_n = \frac{2}{\left(\sinh\frac{n\pi a}{b}\right) b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

Transmission line equation

Telephone equation

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (RC+LG)\frac{\partial v}{\partial t} + RGv \quad \text{—①}$$

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC+LG)\frac{\partial I}{\partial t} + RGI \quad \text{—②}$$

where
R – resistance
L – Inductance
C – Capacitance
G – Leakance

case I

If $L = G = 0$ in equations ① & ②

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}$$

which is called telegraph equation. They are similar to the equation one dimensional heat flow equation.

case II when $R = G = 0$ in the equation ① & ②

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}$$

which is called Radio equation. They are similar to the wave equation

~~case III when R & G are negligible. the transmission line (Telephone line) line become~~

Case III when $L = C = 0$. the equation ① & ② become

$$\frac{\partial^2 v}{\partial x^2} = RGv \quad \& \quad \frac{\partial^2 I}{\partial x^2} = RGI$$

is called equation ~~of~~ for submarine cable.

case IV If R & G are negligible, the transmission lines become

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \quad \& \quad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}$$

Ex 2 A transmission line 1000 km long is initially under steady-state conditions with potential 1300 volts at the sending end ($x=0$) and 1200 volts at receiving at end ($x=1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance (L) and leakance (G) to be negligible find the potential $E(x,t)$.

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Sol^n: Since L and G are negligible we use telegraph equation

$$\frac{\partial^2 E}{\partial x^2} = RC\frac{\partial E}{\partial t} \quad \text{or} \quad \frac{\partial^2 E}{\partial x^2} = \frac{1}{RC}\frac{\partial E}{\partial t} \quad \text{——} (1)$$

Boundary Condition

$$E(0,t) = 1300 \quad \& \quad E(1000,t) = 0 \quad \text{——} (2)$$

Initial ~~ste~~ voltage distribution

$$E_1 = 1300 - \frac{1300-1200}{1000}x = 1300 - 0.1x$$

final voltage distribution

$$E_2 = 1300 - \frac{1300-0}{1000}x = 1300 - 1.3x$$

Initial Condition

$$E(x,0) = E_1 = 1300 - 0.1x$$

} steps.

Sol^n of (1).

$E(x,t) = E_1 + \Sigma$

$$E(x,t) = \overbrace{A_2}^{1300-1.3x} + \sum_{n=1}^{\infty}(a_n\cos px + b_n\sin px)e^{-\left(\frac{p}{RC}\right)^2 t}$$

Applying Boundary Condition

$$E(0,t) = \cancel{1300} = \cancel{1300} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{p}{RC}\right)^2 t}$$

$\Rightarrow a_n = 0$

$$E(0,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{p}{RC}\right)^2 t} = 0$$

$$\Rightarrow a_n = 0$$

Become

$$E(x,t) = 1300 - \underset{1.3x}{\cancel{0.1x}} + \sum_{n=1}^{\infty} b_n \sin px\, e^{-\left(\frac{p}{RC}\right)^2 t}$$

$E(1000, t) = 0,$

$$E(1000,t) = \sum_{n=1}^{\infty} b_n \sin 1000p\, e^{-\left(\frac{p}{RC}\right)^2 t} = 0$$

$$\sin 1000p = 0.$$

$$p = \frac{n\pi}{1000} \quad \forall n \in I$$

$$E(x,t) = \underset{\substack{\downarrow \\ 1300 - 1.3x}}{} \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1000}\, e^{-\left(\frac{n\pi}{1000RC}\right)^2 t}$$

Initial Condition

$$E(x,0) = \cancel{1300} - 0.1x = \cancel{1300} - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1000} \cancel{e^{-\left(\frac{n\pi}{1000RC}\right)^2}}$$

$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1000} \cancel{e^{-\left(\frac{n\pi}{1000RC}\right)^2 t}}$$

$$b_n = \frac{2}{1000}\int_0^{1000} 1.2x \sin\frac{n\pi x}{1000}\, dx$$

$$= \frac{1}{500} \times 1.2 \int_0^{1000} x \sin\frac{n\pi x}{1000}\, dx$$

$$= \frac{1.2}{500}\left[-x\cos\frac{n\pi x}{1000}\times\left(\frac{1000}{n\pi}\right) + \sin\frac{n\pi x}{1000}\times\left(\frac{1000}{n\pi}\right)^2\right]_0^{1000}$$

$$= \frac{1.2 \times \cancel{1000}^{2} \times 1000}{\cancel{500} \times n\pi}\left[-(-1)^n\right]$$

$$= \frac{1200 \times 2}{n\pi}\left[-(-1)^2\right]$$

$$= \frac{2400}{n\pi}(-1)^{n+1}$$

$$E(x,t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi c}{1000}\, e^{-\left(\frac{n\pi}{1000RC}\right)^2 t}.$$

Method of separation of variables :-

Monday, 1 May, 2023 01:34 PM

Let $u(x_1, x_2, x_3 \dots x_n) = X_1(x_1) \cdot X_2(x_2) \cdot X_3(x_3) \cdots X_n(x_n)$

$u(x,y) = X(x) \cdot Y(y)$

$u(x,y,z) = X(x) \cdot Y(y) \cdot Z(z)$

} Note: PDE $\longrightarrow$ ODE.

Q. 1. Use method of separation of variables to solve the equation $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ given that $u(x,0) = 6e^{-3x}$

Soln:- the given equation is $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ —— (1)

Let $u(x,t) = X(x) \cdot T(t)$ —— (2)

So, $\frac{\partial u}{\partial x} = TX'$ & $\frac{\partial u}{\partial t} = XT'$ $\left\{ \begin{array}{l} X' = \frac{dX}{dx} \\ T' = \frac{dT}{dt} \end{array} \right.$

from (1).

$$TX' = 2XT' + XT$$

$$\frac{X'}{X} = \frac{2T'}{T} + 1 = k \quad \text{(say)}.$$

(1) $\frac{X'}{X} = k \Rightarrow X' = kX \Rightarrow \frac{dX}{dx} = kX$

$$\int \frac{dX}{X} = \int k\,dx + \log c_1$$

$$\log X = kx + \log c_1$$

$$X = c_1 e^{kx}$$

$\left(\frac{k-1}{2}\right)t$

$$X = c_1 e$$

and (2) $$T = c_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$u(x,t) = X\cdot T = c_1 e^{kx} \cdot c_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$= c_1 c_2 e^{kx + \left(\frac{k-1}{2}\right)t}$$

Initial Condition

$$u(x,0) = 6e^{-3x}$$

$$u(x,0) = 6e^{-3x} = c_1 c_2 e^{kx}$$

$$\Rightarrow \quad k = -3 \quad \& \quad c_1 c_2 = 6$$

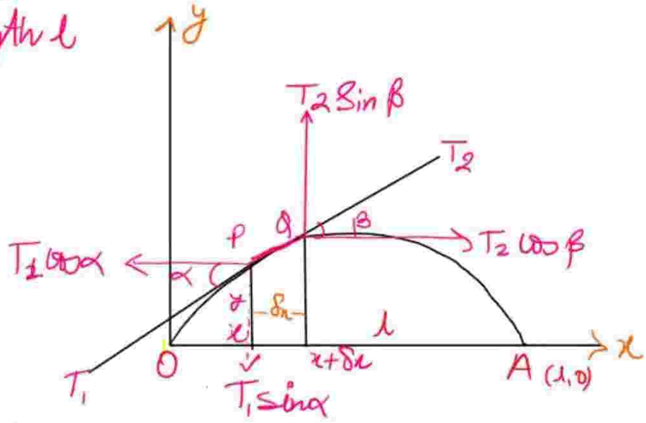
Hence the required solution is

$$\boxed{u(x,t) = 6\, e^{-3x-2t}}$$

One Dimensional wave equation $\left(\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\right)$ —

Consider a uniform elastic string of length l stretched tightly between O & A

Taking O origin & OA on the x-axis, perpendicular to O as the y-axis



we then distort the string and at some instant call it $t=0$ we release it and allow it to vibrate.

Physical Assumption . (a) $\frac{m}{l}$ = constant

(b) Tension $T >> W = mg$

(c) Motion is contraint in xy-plane

The tension T_1 at P and T_2 at Q.

Since, there is no motion in the horizontal direction

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant)} \quad \text{——} (1)$$

By Newton second law of motion

$$Ma = F$$

$$m\delta x \frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \quad \text{——} (2)$$

$\frac{(1)}{(2)}$

$$\Rightarrow \quad \frac{m\delta x}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta x}\left(\tan\beta - \tan\alpha\right)$$

$$= \frac{T}{m}\left(\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\quad}\right)$$

$$= \frac{T}{m}\left(\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right)$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}}$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \qquad \text{where } c^2 = \frac{T}{m}$$

which is called one - Dim - wave equation

<u>Boundary Condition</u>

$$y(0,t) = 0 \qquad \& \qquad y(l,t) = 0 \qquad t \geq 0$$

<u>Initial Conditions</u>

$$y(x,0) = f(x), \qquad y_t(x,0) = 0 \qquad 0 \leq x \leq l$$

Solution of one dim. wave equation:-

we have,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{—— (1)}$$

let $\quad y(x,t) = X(x) \cdot T(t)$

$$XT'' = c^2 T X''$$

$$\boxed{\frac{T''}{c^2 T} = \frac{X''}{X} = K}$$

$$T'' - c^2 K = 0 \qquad \& \qquad X'' - KX = 0 \qquad \text{—— (2)}$$

(1) when K is positive, $\ni K = p^2$

$$X'' - p^2 X = 0$$

A.E.

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

Similary $T = c_3 e^{cpt} + c_4 e^{-pct}$

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-pct}) \quad \text{---(A)}$$

(2) when k is Negative $k = -p^2$

$$X'' + p^2 X = 0 \quad \& \quad T'' + c^2 p^2 T = 0$$

A.E $m^2 + p^2 = 0 \Rightarrow m = \pm ip$

$X = c_5 \cos px + c_6 \sin px$, Similarly $T = c_7 \cos pct + c_8 \sin pct$

Thus, $y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)$ ---(B)

Case III when k is zero, $k = 0$

$X'' = 0$, & $T'' = 0$ from (2).

$X' = c_9$

$X = c_9 x + c_{10}$ Similarly $T = c_{11} t + c_{12}$

Thus, $y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12})$ ---(C)

Here, (A) (B) & (C) are solution of one-dim. wave equation

The y must be a periodic function of x and t. therefore the solution must be envolving trigo. fuction

So,

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct) \quad \text{---(3)}$$

Applying Boundary Condition

$y(0,t)=0$, $y(l,t)=0$

$$y(0,t)=c_5\,(c_7 \cos cpt + c_8 \sin cpt)=0$$

$$\Rightarrow c_5=0$$

Become (3)

$$y(x,t)= c_6 \sin px\,(c_7 \cos pct + c_8 \sin pct) \quad \text{——— (4)}$$

$$y(l,t)= c_6 \sin pl\,(c_7 \cos pct + c_8 \sin pct)=0$$

$c_6 \neq 0$, $\sin pl = 0 \Rightarrow \sin pl = \sin n\pi$ where $n \in I$

$$p=\frac{n\pi}{l}$$

from (4) $y(x,t)= c_6 \sin \frac{n\pi x}{l}\left(c_7 \cos \frac{n\pi ct}{l} + c_8 \sin \frac{n\pi ct}{l}\right)$

$$y(x,t)=\left(c_6 c_7 \cos \frac{n\pi ct}{l} + c_6 c_8 \sin \frac{n\pi ct}{l}\right)\sin \frac{n\pi x}{l}$$

$$y(x,t)=\left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

where $a_n = c_6 c_7$ & $b_n = c_6 c_8$

The general solution is

$$y(x,t)=\sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right)\sin \frac{n\pi x}{l}$$

Applying initial Conditions

$y(x,0)=f(x)$

$$f(x)=\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}, \quad \text{where } a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

which is represent fourier series.

Now

Now
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left[a_n \left(-\sin \frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l} + b_n \cos \frac{n\pi ct}{l} \times \left(\frac{n\pi c}{l}\right) \right] \times \sin \frac{n\pi x}{l}$$

$$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n \left(-a_n \sin \frac{n\pi ct}{l} + b_n \cos \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n(b_n) \sin \frac{n\pi x}{l} = 0$$

$$\Rightarrow b_n = 0$$

Hence the required solution is

$$\boxed{y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \cdot \sin \frac{n\pi x}{l}}$$

where $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$

2017, 2018, 2019,

Q. A string is stretched and fastened to two points l apart. motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it released at time $t=0$. Show that displacement of any point at distance x form one end at time t is given by

$$\boxed{y(x,t) = A \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}}$$

Sol (i) the one dim. wave equation $\frac{\partial^2 y}{\ldots}$ ...

Sol (1) The one dim. wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —(1)

Boundary Condition.

$y(0,t)=0$, & $y(l,t)=0$

Initial Condition

$y(x,0) = A \sin\frac{\pi x}{l}$ (Given)

$y_t(x,0) = 0$ (since the initial velocity is zero)

let $y(x,t) = X(x) \cdot T(t)$

$XT'' = c^2 TX''$

$$\boxed{\frac{T''}{c^2 T} = \frac{X''}{X} = k}$$

$T'' - c^2 K = 0$ & $X'' - kX = 0$ —(2)

(1) when k is positive, $\Rightarrow K = p^2$

$X'' - p^2 X = 0$

A.E.

$m^2 - p^2 = 0 \Rightarrow m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

Similary $T = c_3 e^{cpt} + c_4 e^{-pct}$

$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-pct})$ —(A)

(2) when k is Negative $K = -p^2$

(2) when k is Negative $k = -p^2$

$$X'' + p^2 X = 0 \quad \& \quad T'' + c^2p^2 T = 0$$

A.E $m^2 + p^2 = 0 \Rightarrow m = \pm ip$

$X = c_5 \cos px + c_6 \sin px$, Similarly $T = c_7 \cos pct + c_8 \sin pct$

Thus, $y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)$ — (B)

Case III when k is zero, $k = 0$

$X'' = 0$, & $T'' = 0$ from (2).

$X' = c_9$

$X = c_9 x + c_{10}$ Similarly $T = c_{11} t + c_{12}$

Thus, $y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12})$ — (C)

Here, (A) (B) & (C) are solution of one-dim. wave equation

The y must be a periodic function of x and t. therefore the solution must be envolving trigo. function

so,

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct) \quad \text{— (3)}$$

Applying Boundary Condition

$y(0,t) = 0$, $y(l,t) = 0$

$$y(0,t) = c_5 (c_7 \cos cpt + c_8 \sin cpt) = 0$$

$$\Rightarrow c_5 = 0$$

Become (3)

$$y(x,t) = c_6 \sin px (c_7 \cos pct + c_8 \sin pct) \quad \text{— (4)}$$

$$y(l,t) = c_6 \sin pl\,(c_7 \cos pct + c_8 \sin pct) = 0$$

$c_6 \neq 0$, $\sin pl = 0 \Rightarrow \sin pl = \sin n\pi$ where $n \in I$

$$p = \frac{n\pi}{l}$$

from (4) $$y(x,t) = c_6 \sin \frac{n\pi x}{l} \left(c_7 \cos \frac{n\pi ct}{l} + c_8 \sin \frac{n\pi ct}{l} \right)$$

$$y(x,t) = \left(c_6 c_7 \cos \frac{n\pi ct}{l} + c_6 c_8 \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Apply initial conditions, $y(x,0) = A \sin \frac{\pi x}{l}$

$$y_t(x,0) = 0$$

$$\frac{\partial y}{\partial t} = \left(-c_6 c_7 \sin \frac{n\pi ct}{l} \times \frac{n\pi c}{l} + c_6 c_8 \cos \frac{n\pi ct}{l} \times \frac{n\pi c}{l} \right) \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi c}{l} c_6 c_8 \sin \frac{n\pi x}{l} = 0 \Rightarrow c_8 = 0$$

$$y(x,t) = c_6 c_7 \cos \frac{n\pi ct}{l} \times \sin \frac{n\pi x}{l}$$

Next

$$y(x,0) = A \sin \frac{\pi x}{l} = c_6 c_7 \sin \frac{n\pi x}{l}$$

thus, $c_6 c_7 = A$, $n = 1$

Therefore $$y(x,t) = A \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l}$$

1. Use the method of separation of variables to solve the equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$
2. Use the method of separation of variables to solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$.
3. Solve the following equation by the method of separation of variables $\frac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos\,x$
4. given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$.

5. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A\sin\,\frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = A\sin\,\frac{\pi x}{l}\cos\,\frac{\pi c t}{l}$$

6. Show how the wave equation $c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between $x = 0$ and $x = l$ are given by $y = f(x)$ and $\frac{\partial y}{\partial t} = g(x)$, determine the constants in the series solution.
7. A string is stretched between two fixed points $(0,0)$ and $(l, 0)$ and released at rest from the initial deflection given by

$$f(x) = \begin{cases} \left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2} \\ \left(\frac{2k}{l}\right)(l - x), & \frac{l}{2} < x < l \end{cases}$$

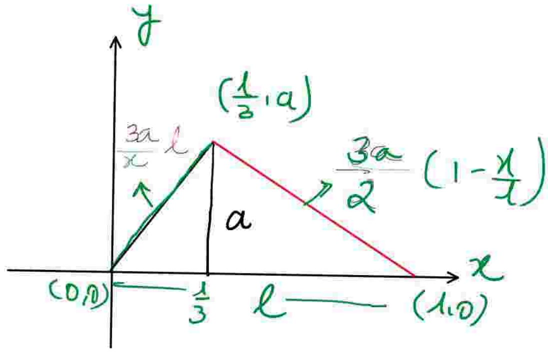
Find the deflection of the string at any time.

8. A tightly stretched violin string of length l and fixed at both ends is plucked $atx = \frac{l}{3}$ and assumes initially the shape of a triangle of height a. Find the displacement y at iny distance x and any time t after the string is released from rest.
9. The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.
10. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $\sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$.

Lecture 3

Monday, 1 May, 2023 10:54 PM

Q. 3. A lightly stretched string of length l and fixed at both ends in plucked at $x = \frac{l}{3}$ and assumes initially the shape of triangle of hight a. find the displacement y at any distance x and any t after the string is released from rest



Sol^n The given wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Boundary Condition

$$y(0,t) = y(l,t) = 0 \qquad \forall\, t \geq 0$$

Initial Condition

$$y(x,0) = f(x) = \begin{cases} \frac{3ax}{l} & ; 0 < x < \frac{l}{3} \\ \frac{3a}{2}\left(1 - \frac{x}{l}\right) & \frac{l}{3} < x < l \end{cases}$$

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

{ step :

Therefore must general solution of ① is

...fore most general solution of (1) is

$$y(x,t)=\sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left\{\int_0^{l/3}\frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx+\int_{l/3}^{l}\frac{3a}{2}\left(1-\frac{x}{l}\right)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$=\frac{6a}{l^2}\left\{\left[x\left(-\cos\frac{n\pi x}{l}\right)\times\frac{l}{n\pi}+\sin\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)^2\right]_0^{l/3}\right\}$$

$$\frac{3a}{l}\left[\left(1-\frac{x}{l}\right)\left(-\cos\frac{n\pi x}{l}\right)\times\frac{l}{n\pi}-\frac{1}{l}\sin\frac{n\pi x}{l}\times\left(\frac{l}{n\pi}\right)^2\right]_{l/3}^{l}$$

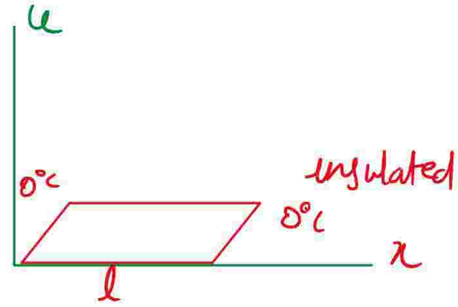
$$=\frac{6a}{l^2}\left[-\frac{l}{3}\cos\frac{n\pi}{3}\times\frac{l}{n\pi}+\frac{l^2}{(n\pi)^2}\sin\frac{n\pi}{3}\right]$$

$$+\frac{3a}{l}\Big[$$

One - Dimensional Heat Equation

Tuesday, 2 May, 2023 11:50 AM

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$



where $c^2 = \frac{K}{s\rho}$ is called diffusivity of material of the bar.

Boundary Condition

$$u(0,t) = u(l,t) = 0$$

Initial Condition

$$u(x,0) = f(x).$$

Solution of Heat equation

The given equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ——— (1)

Let $u = X(x) \cdot T(t)$

$$XT' = c^2 T X''$$

$$\frac{X''}{X} = \frac{T'}{c^2 T} = K \quad \text{——— (3)}$$

Case I when K is positive, $K = p^2$

$$\frac{X''}{X} = p^2 \Rightarrow \quad X'' - p^2 X = 0$$

A.E $m^2 - p^2 = 0$

A.E $\quad m^2 - p^2 = 0$

$m = \pm p$

$C.f = c_1 e^{px} + c_2 e^{-px}$, & $P.I = 0$

$X = c_1 e^{px} + c_2 e^{-px}$

and $\quad \frac{T'}{c^2 T} = p^2 \Rightarrow T' = c^2 p^2 T \Rightarrow \int \frac{dT}{T} = \int c^2 p^2 dt + \log c_3$

$\log T = c^2 p^2 t + \log c_3$

$T = c_3 e^{c^2 p^2 t}$

$u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{c^2 p^2 t}) \longleftarrow$ (A)

Case II $\quad$ when k is Negative, $\quad k = -p^2$

from (3)

$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$

A.E. $\quad m^2 + p^2 = 0$

$m = \pm ip$

$C.f = (c_4 \cos px + c_5 \sin px)$ & $P.I = 0$

$X = (c_4 \cos px + c_5 \sin px)$

and $\quad \frac{T'}{T} = -p^2 c^2 \Rightarrow \int \frac{dT}{T} = \int -p^2 c^2 dt + \log c_6$

$T = c_6 e^{-p^2 c^2 t}$

$$u(x,t) = (c_4 \cos px + c_5 \sin px)(c_6 e^{-p^2c^2 t}) \quad \text{—— (B)}$$

Case III when $k=0$

from (3)

$$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \qquad \frac{T'}{T} = 0$$

$$X' = c_7 \qquad T' = 0$$

$$X = c_7 x + c_8 \qquad T = c_9$$

Hence $u(x,t) = (c_7 x + c_8)\, c_9$ —— (C)

Here (A), (B) & (C) are three solutions of Heat equation

But u decreased at time increase.

$\Rightarrow t \to \infty$, $u(x,t) = 0$

Thus, $u(x,t) = (c_4 \cos px + c_5 \sin px)\, c_6 e^{-p^2c^2 t}$ —— (4)

is required solution.

Boundary condition

$$u(0,t) = 0 \quad \& \quad u(l,t) = 0$$

$$u(0,t) = c_4 c_6 e^{-p^2c^2 t} = 0$$

$$\Rightarrow c_4 = 0$$

from (4)

$$u(x,t) = c_5 \sin px\, c_6 e^{-p^2c^2 t}$$

& $u(l,t) = 0$

&

$$u(l,t)=0$$

$$u(l,t) = c_5 c_6 \sin pl \; e^{-p^2c^2t} = 0$$

$$\sin pl = 0 \Rightarrow \quad p = \frac{n\pi}{l} \;\; \forall n \in I$$

$$u(x,t) = c_5 c_6 \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

The must general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

Initial Condition

$$u(x,0) = f(x)$$

$$\Rightarrow \quad u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

which is represent fourier series, so $a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Hence the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

where $a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.

1. A rod of length l with insulated sides is initially at a uniform temperature u_o. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x,t)$.

2. Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

3. An insulated rod of the length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.

   Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

4. The ends A and B of a rod of length 20cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function $u(x, t)$.

Solⁿ 1. the Heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

Boundary Condition (ATQ).

$$u(0,t) = u(l,t) = 0 \quad —— ②$$

Initial Condition $u(x,0) = f(x) = u_0$ —— ③

Solⁿ of ① is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)c_3 e^{-c^2p^2t} \quad —— ④$$

Boundary Condition

$$u(0,t) = c_1 c_3 e^{-c^2p^2t} = 0 \Rightarrow c_1 = 0$$

from ④.

$$u(x,t) = c_2 \sin px \, c_3 e^{-c^2p^2t}$$

$$u(l,t) = c_2 \sin pl \, c_3 e^{-c^2p^2t} = 0$$

$$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l} \quad \forall n \in I$$

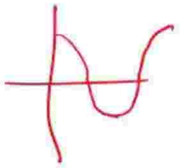
$$u(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-c^2\left(\frac{n\pi}{l}\right)^2 t}.$$

Most general solution $u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{c^2 n^2\pi^2}{l^2} \cdot t}$

Initial Condition

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

By fourier series $a_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$



$$= \frac{2u_0}{l}\left[-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l$$

$$= \frac{2u_0}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

Hence

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{c^2p^2\pi^2 t}{l^2}}$$

$$\boxed{= \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2p^2\pi^2 t}{l^2}}}$$

The ends A and B of a rod of length 20cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°Cand 60°C respectively. Find the temperature distribution function $u(x,t)$.

Sol^{n} Equation of Heat $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ ——— (1)

Initial temperature distribution in the rod is $\left\{ u_1 = \alpha + \left(\frac{\beta-\alpha}{l}\right)x \right.$

$$u_1 = 30 + \left(\frac{80-30}{20}\right)x$$

$$= 30 + \frac{5}{2}x$$

final temperature distribution in the rod is

$$u_2 = 40 + \left(\frac{60-40}{20}\right)x = 40 + x$$

Thus, Boundary Condition (In steady state)

$$u(0,t) = 40 \qquad \& \quad u(20, t) = 60 \quad \text{——— (2)}$$

and Initial Condition is

$$u(x,0) = 30 + \frac{5}{2}x \quad \text{——— (3)}$$

Step लिखा होना ?

therefore solution of (1) is

$$u(x,t) = u_2 + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2c^2t}$$

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2c^2t} \quad \text{——— (4)}$$

Boundary Condition

$$u(0,t) = 40$$

$$\Rightarrow u(0,t) = \cancel{40} = \cancel{40} + 0 + \sum_{n=1}^{\infty} a_n e^{-p^2c^2t}$$

$$\sum_{n=1}^{\infty} a_n e^{-p^2c^2t} = 0$$

$$\Rightarrow a_n = 0$$

equation (4)

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin px \; e^{-p^2c^2t} \quad \text{——— (6)}$$

$$u(20, t) = \cancel{60} = \cancel{40+20} + \sum^{\infty} b_n \sin p(20) \cdot e^{-p^2c^2t}$$

Q $u(20, t) = \cancel{60} = \cancel{40+20} + \sum_{n=1}^{\infty} b_n \sin p(20) \cdot e^{-p^2c^2t}$

$$\sum_{n=1}^{\infty} b_n \sin 20p \cdot e^{-p^2c^2t} = 0$$

$$\Rightarrow \sin 20p = 0$$

$$p = \frac{n\pi}{20} \quad \forall\, n \in I.$$

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \cdot e^{-\left(\frac{n\pi}{20}\right)^2 \cdot c^2 \cdot t}$$

Initial condition

$$u(x,0) = 30 + \frac{5}{2}x$$

$$30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

$$\underline{\frac{3}{2}x - 10} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

by fourier series.

$$b_n = \frac{2}{20}\int_0^{20} \left(\frac{3}{2}x - 10\right) \cdot \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}x - 10\right)\left(-\cos \frac{n\pi x}{20}\right) \cdot \left(\frac{20}{n\pi}\right) + \frac{3}{2} \sin \frac{n\pi x}{20} \left(\frac{20}{n\pi}\right)^2\right]_0^{20}$$

$$= \frac{1}{10}\left[\frac{-20 \times 20}{n\pi} \cos n\pi - \frac{200}{n\pi}\right]$$

$$= \frac{-20}{n\pi}\left(2(-1)^n + 1\right)$$

Hence

$$\boxed{u(x,t) = 40 + x - \frac{20}{n\pi}\sum_{n=1}^{\infty}\left(\frac{2(-1)^n + 1}{n}\right)\sin\left(\frac{n\pi x}{20}\right)\cdot e^{-\left(\frac{n\pi c}{20}\right)^2 \cdot t}}$$

Example 8. *Solve:* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ *under the conditions*

(*i*) $u \neq \infty$ *if* $t \to \infty$

(*ii*) $\dfrac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$

(*iii*) $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$. (U.P.T.U. 2015)

Two dimensional Case:-

Wednesday, 3 May, 2023 10:45 AM

Two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \frac{T}{m}$

Two dimensional Heat flow equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \frac{K}{s\rho}$ is called diffusivity of the material.

In case of steady $\left(\frac{\partial u}{\partial t} = 0\right)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

which is called Laplace equation.

Solution of Laplace equation as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Let $u(x,y) = X \cdot Y$ —— (2)

Using (2),

$$Y X'' + X Y'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = K \quad (\text{say}). \quad \text{--- (2)}$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = K \quad (\text{say}) \text{ ——— } (2)$$

Case I when $K = p^2$

$$\frac{X''}{X} = p^2 \Rightarrow X'' - p^2 X = 0$$

A.E. $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$C.F = (c_1 e^{px} + c_2 e^{-px}), \quad P.I = 0$$

Similarly

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$Y = c_3 \cos py + c_4 \sin py$$

Thus, $u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$ ——— (A)

Case II $K = -p^2$

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \text{ ——— } (B)$$

Case III $k = 0$

$$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12}) \text{ ——— } (C)$$

Here, (A), (B) & (C) are three solution of (1).
But we have to choose that solution which is consistent with Physical nature of the problem and boundary condition
Hence (B) is required solution

$$u(x,y) = (C_5\cos px + C_6\sin px)\,(C_7 e^{py} + C_8 e^{-py}).$$

Q.1. Use seperation of variables method to solve the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Subjet to boundary condition

$$u(0,y) = u(l,y) = u(x,0) = 0$$

& $u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$.

Sol^n The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ——— (1)

Step ?

for $k = p^2$ $\quad u(x,y) = (C_1 e^{px} + C_2 e^{-px})\,(C_3\cos py + C_4\sin py)$ ——— (A)

for $k = 0$ $\quad u(x,y) = (C_9 x + C_{10})\,(C_{11} y + C_{12})$ ——— (B)

for $k = -p^2$ $\quad u(x,y) = (C_5\cos px + C_6\sin px)\,(C_7 e^{py} + C_8 e^{-py})$ ——— (C)

Applying Boundary condition in (A)

$u(x,0) = 0$,

$$u(x,0) = (C_1 e^{px} + C_2 e^{-px})\,C_3 = 0$$

$$\Rightarrow C_3 = 0$$

from (A) $\quad u(x,y) = (C_1 e^{px} + C_2 e^{-px})\,C_4\sin py$ ——— (2)

$$u(0,y) = (C_1 + C_2)\,C_4 \sin py.$$

$$u(0,y) = (c_1 + c_2) c_4 \sin px$$

$$\Rightarrow \quad c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

from (2) $\quad u(x,y) = c_1 (e^{px} - e^{-px}) c_4 \sin pxy$

$u(l,y) = 0,$

$$u(l,y) = c_1 (e^{pl} - e^{-pl}) c_4 \sin py = 0$$

$$\Rightarrow \quad c_1 = 0 \quad \text{or} \quad c_4 = 0$$

Thus, $\quad u(x,y) = 0 \quad$ which is impossible.

Applying Boundary Condition in (B).

$$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{——} \quad (3)$$

$u(0,y) = 0$

$$u(0,y) = (c_9 \times 0 + c_{10})(c_{11} y + c_{12}) = 0$$

$$\Rightarrow \quad c_{10} = 0$$

from (3)

$$u(x,y) = c_9 x (c_{11} y + c_{12}) \quad \text{——} \quad (4)$$

$u(x,0) = 0,$

$$u(x,0) = c_9 x (c_{11} \times 0 + c_{12}) = 0$$

$$\Rightarrow \quad c_{12} = 0$$

from (4).

$$u(x,y) = c_9 c_{11}\ xy$$

$$u(l,y) = c_9 c_{11}\ ly = 0$$

$$c_9 = 0 \text{ or } c_{11} = 0$$

Thus, $u(x,y) = 0$ which impossible

Applying boundary condition in (c).

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad — (5)$$

$u(0,y) = 0,$

$$u(0,y) = c_5 \left(c_7 e^{py} + c_8 e^{-py} \right) = 0$$

$$\Rightarrow c_5 = 0$$

from (5)

$$u(x,y) = c_6 \sin px \left(c_7 e^{py} + c_8 e^{-py} \right) \quad — (6)$$

$u(x,0) = 0,$

$$u(x,0) = c_6 \sin px \left(c_7 + c_8 \right) = 0 \quad \Rightarrow \quad c_7 + c_8 = 0$$

$$c_7 = -c_8$$

from (6)

$$u(x,y) = c_6 c_7 \sin px \left(e^{py} - e^{-py} \right) \quad — (7)$$

$u(l, y) = 0$

$$u(l,y) = c_6 c_7 \sin pl \left(e^{py} - e^{-py}\right) = 0$$

$$\sin pl = 0$$
$$p = \frac{n\pi}{l} \quad \forall n \in I$$

from (7)

$$u(x,y) = 2c_6 c_7 \sin\left(\frac{n\pi x}{l}\right) \left(\frac{e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}}{2}\right)$$

$$u(x,y) = 2c_6 c_7 \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi y}{l}\right) \quad \text{——— (8)}$$

and

$$u(x,a) = \sin \frac{n\pi x}{l}$$

$$u(x,a) = \cancel{\sin \frac{n\pi x}{l}} = 2c_6 c_7 \cancel{\sin \frac{n\pi x}{l}} \cdot \sinh\left(\frac{n\pi a}{l}\right)$$

$$2c_6 c_7 = \frac{1}{\sinh\left(\frac{n\pi a}{l}\right)}$$

from (8)

$$u(x,y) = 2c_6 c_7 \sin \frac{n\pi x}{l} \; \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)}.$$

2. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.